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MULTIPLY PERFECT NUMBERS OF THREE DIFFERENT PRIMES[*]

By R. D. Carmichael

The object of the present paper is to complete the investigation of the existence of multiply perfect numbers of three different primes.[†]

J. Westlund has shown[‡] that the only such numbers of multiplicity 3 of the form $p_1^{a_1} p_2^{a_2} p_3$, where $p_1 < p_2 < p_3$, p_1, p_2, p_3 being different primes, are $2^3 \cdot 3 \cdot 5$ and $2^5 \cdot 3 \cdot 7$. I have shown [§] that there is none of multiplicity 3 of the form $p_1^{a_1} p_2^{a_2} p_3^{2a_3}$, where p_1, p_2, p_3 are as before; and also[||] that there exist no multiply perfect odd numbers containing (only) three different primes.

It is easily shown that the multiplicity of the numbers under consideration is either 2 or 3. The following proposition completes the investigation of the latter case:

There exist no multiply perfect numbers of multiplicity 3 *of the form* $p_1^{a_1} p_2^{a_2} p_3^{2a_3+1}$, *where* p_1, p_2, p_3 *are different primes and* $p_1 < p_2 < p_3$ *and* $a_3 > 0$.

For this case, we should have, corresponding to equation (3) of a previous note (see Annals, *l. c.* p. 153), the following equation:

$$(1) \qquad p_3^{2a_3+1} + p_3^{2a_3} + \cdots + p_3 + 1 = \frac{p_3^{2a_3+2} - 1}{p_3 - 1} = 2^r \cdot 3^s,$$

where the values of r and s are to be considered, and $r > 0$ and $s \geqq 0$. Since $p_3 > 3$, it is of one of the forms $6n \pm 1$.

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[*] Presented to the American Mathematical Society, September 3, 1906.

[†] It can be shown without difficulty that there exists no multiply perfect number which is the power of a prime; also, that every multiply perfect number, containing only two different primes, is of the form $2^{n-1}(2^n - 1)$, where $2^n - 1$ is prime; and, moreover, that every such number is a perfect number of multiplicity 2. This paper therefore completes the investigation for numbers of fewer than 4 different primes.

[‡] Annals of Mathematics, vol. 2, p. 172; July, 1901.

[§] Annals of Mathematics, vol. 7, p. 153; April, 1906.

[||] American Mathematical Monthly, vol. 13, p. 35; 1906.

CASE 1. Let $p_3 = 6n + 1$. Equation (1) now becomes

$$\frac{(6n+1)^{2a_3+2} - 1}{6n} = 2^r \cdot 3^s. \tag{2}$$

Therefore,

$$(6n)^{2a_3+1} + \cdots + (a_3+1)(2a_3+1)6n + 2(a_3+1) = 2^r \cdot 3^s. \tag{3}$$

Now, let $a_3 + 1 = 2^\lambda \cdot 3^\mu \cdot t$, where t is not divisible by 2 or 3. Every term of equation (3), except the last two, is divisible by $2^{\lambda+2} \cdot 3^{\mu+1}$. Dividing through by $2^{\lambda+1} \;\; 3^\mu$, we may write (q an integer quotient)

$$6q + t\,[3n\,(2^{\lambda+1} \cdot 3^\mu \cdot t - 1) + 1] = 2^{r-\lambda-1} \cdot 3^{s-\mu}, \tag{4}$$

where $2a_3 + 1$ is replaced by its value $2^{\lambda+1} \cdot 3^\mu \cdot t - 1$. The left member of (4) is not divisible by 3. Hence, $s - \mu = 0$. q is not less than 1; for it is evidently positive and not zero. Also, $2^{\lambda+1} \cdot 3^\mu \cdot t - 1 \geqq 3$ and $t \geqq 1$. Hence,

$$2^{r-\lambda-1} \geqq 9n + 7, \quad n \geqq 1. \tag{5}$$

Since t is odd, the conditions of (4) and (5) can be satisfied only when n is odd. In this case $6n + 1$ may be written in the form $4m - 1$. Substituting this value in equation (2) and reducing, we have

$$\begin{aligned}(4m)^{2a_3+2} + \cdots + (a_3+1)(2a_3+1)(4m)^2 - 2(a_3+1)\,4m \\ = 2^{r+1} \cdot 3^s(2m-1).\end{aligned} \tag{6}$$

Now, let $m = 2^x \cdot l$, l an odd factor. Then $2^{\lambda+x+3}$ divides every term but the last of the first member of (6) with an even quotient, and the last with an odd quotient; hence it is the highest power of 2 in that member. Then we must have $\lambda + x + 3 = r + 1$; or, $r - \lambda - 1 = x + 1$. Again, since $6n + 1 = 4m - 1$, $9n + 7 = 6m + 4$. These values substituted in (5) give

$$2^{x+1} \geqq 6m + 4 = 2^{x+1} \cdot 3l + 4,$$

a clearly impossible inequality. Hence, case I yields no numbers of the type here considered.

CASE II. Let $p_3 = 6n - 1$. Substituting in equation (1), using the fraction for first member, we have by reduction

$$\begin{aligned}(6n)2^{a_3+2} + \cdots + (a_3+1)(2a_3+1)(6n)^2 - 2(a_3+1)6n \\ = 2^{r+1} \cdot 3^s\,(3n-1).\end{aligned} \tag{7}$$

Again put $a_3 + 1 = 2^\lambda \cdot 3^\mu \cdot t$; also let $n = 2^x \cdot 3^y \cdot l$, where l is not divisible by 2 or 3. Here $2^{\lambda+3x+3} \cdot 3^{\mu+3y+2}$ divides every term of the first member of

(7) except the last two. Therefore, dividing through by $2^{\lambda+x+2}\cdot 3^{\mu+y+1}$, we may write ($q$ being an integer, which is readily shown to be positive and > 0)

$$(8)\qquad 2^{2x+1}\cdot 3^{2y+1}q + lt[2^x\cdot 3^{y+1}l(2^{\lambda+1}\cdot 3^{\mu}t-1)-1] = 2^{r-\lambda-x-1}\cdot 3^{s-\mu-y-1}(2^x\cdot 3^{y+1}l-1).$$

The left member is not divisible by 3; neither is the quantity $(2^x\cdot 3^{y+1}l-1)$ in the second member. Hence,

$$s-\mu-y-1=0.$$

If $x\neq 0$, that is if n is even, we have $6n-1=4m-1$, where $m = 2^{x-1}\cdot 3^{y+1}$. Putting $4m-1$ for $6n-1$ as the value of p_3 in equation (1) we reproduce equation (6). With the present value of m, $2^{\lambda+x+2}$ divides every term except the last of the first member of this equation with an even quotient, and the last with an odd quotient; hence it is the highest power of 2 in that member. Then we must have $\lambda+x+2=r+1$; or $r-\lambda-x-1 = 0$. This condition and $s-\mu-y-1=0$ above found reduces equation (8) to the form

$$(8')\quad 2^{2x+1}\cdot 3^{2y+1}\cdot q + lt[2^x\cdot 3^{y+1}l(2^{\lambda+1}\cdot 3^{\mu}t-1)-1] = 2^x\cdot 3^{y+1}l-1,$$

which is clearly impossible, since q is positive and $2^{\lambda+1}\cdot 3^{\mu}t-1 = 2a_3 + 1 \geqq 3$. This absurdity is the result of the supposition that n is even.

If n is odd, $6n-1$ may be written in the form $4m+1$. Substituting this value in equation (1) and reducing, we have

$$(9)\qquad (4m)^{2a_3+1}+\cdots+(a_3+1)(2a_3+1)4m+2(a_3+1)=2^r\cdot 3^s.$$

Here $2^{\lambda+1}$ divides every term but the last of the first member of (9) with an even quotient, and the last with an odd quotient; hence, it is the highest power of 2 in that member. Therefore $\lambda+1=r$. Substituting this value in equation (8), remembering that $s-\mu-y-1=0$, and that now $x=0$, we have

$$(8'')\qquad 2\cdot 3^{2y+1}q + lt[3^{y+1}l\,(2^{\lambda+1}\cdot 3^{\mu}t-1)-1] = 3^{y+1}l-1,$$

which is clearly impossible. This case therefore yields no numbers of the type under consideration. Hence the proposition as announced above.

We propose now to prove the following theorem:

There exist no multiply perfect even numbers containing (only) three different primes and having a multiplicity of 2.

Let the form be $2^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$. From the definition of multiply perfect numbers we readily have (by means of the formula for the sum of the divisors of a number),

$$(10) \qquad 2 = \frac{2^{\alpha_1+1}-1}{2^{\alpha_1}} \cdot \frac{p_2^{\alpha_2} + p_2^{\alpha_2-1} + \cdots + p_2 + 1}{p_2^{\alpha_2}} \cdot \frac{p_3^{\alpha_3} + \cdots + p_3 + 1}{p_3^{\alpha_3}}.$$

Either α_2 or α_3 is odd, as otherwise all the numerators would be odd, which is inadmissible, the left member being even. Take α_3 odd. (It is to be noticed that in this case no assumption has been made with regard to any relations between p_2 and p_3; and hence to take specifically α_3 odd causes no loss of generality.) Equation (10) may now be written

$$(11) \qquad 2 = \frac{2^{\alpha_1+1}-1}{2^{\alpha_1}} \cdot \frac{p_2^{\alpha_2} + \cdots + p_2 + 1}{p_2^{\alpha_2}} \cdot \frac{(p_3+1)(1+p_3^2+p_3^4+\cdots+p_3^{\alpha_3-1})}{p_3^{\alpha_3}}.$$

Suppose, first, $p_3 + 1$ a power of 2, and let $p_3 + 1 = 2^{\mu}$. If the last factor in the numerator above has an odd number (> 1) of terms, it can always be separated into two different odd factors, of the same degree in p_3 and each prime to p_3; thus, in the case supposed, $\alpha_3 - 1 = 4x$, x an integer; the factors are therefore

$$(1 + p_3 + p_3^2 + p_3^3 + \cdots + p_3^{2x})(1 - p_3 + p_3^2 - p_3^3 + \cdots + p_3^{2x}) = 1 + p_3^2 + p_3^4 + \cdots + p_3^{4x}.$$

These two odd factors, being each prime to p_3, introduce at least two odd primes different from p_3, unless they are both powers of the same prime. First suppose the latter case. Let r be a prime different from p_3, and let

$$1 + p_3 + p_3^2 + \cdots + p_3^{2x} = r^{\alpha}; \quad 1 - p_3 + p_3^2 - p_3^3 + \cdots + p_3^{2x} = r^{\beta},$$

where $\beta < \alpha$. Since each quantity is a power of the prime r, their difference is divisible by the less. Hence,

$$\frac{2p_3(1 + p_3^2 + p_3^4 + \cdots + p_3^{2x-2})}{1 - p_3 + p_3^2 - p_3^3 + \cdots + p_3^{2x}} = \text{an integer}.$$

Now $2p_3$ is prime to the denominator; hence the quantity in parenthesis in the numerator must be a multiple of the denominator. Call the quantity in parenthesis N and the denominator D. Then

$$p_3^{2x} - p_3^{2x-1} < D.$$

Also,

$$N < p_3^{2x-1} = \frac{(p_3^2 - 1)N + 1}{p_3} = p_3 N - \frac{N-1}{p_3};$$

for $p_3 \geqq 3$. But

$$p_3^{2x-1} < p_3^{2x-1}(p_3 - 1) = p_3^{2x} - p_3^{2x-1}.$$

Hence, $N < D$; therefore N is not a multiple of D. Hence $1 + p_3 + p_3^2 + \cdots + p_3^{2x}$ and $1 - p_3 + p_3^2 - p_3^3 + \cdots + p_3^{2x}$ are not powers of the same prime. These two factors therefore introduce (at least) two odd primes different from p_3, which is inadmissible. Hence, only two conclusions are now possible, namely: the last factor in the numerator of (11) contains only one term, in which case $a_3 = 1$; or, it contains an even number of terms.

Take first the case when there is only one term in the expression. That term is unity. For this case equation (10) may be written

$$(12) \qquad 2 = \frac{2^{a_1+1} - 1}{2^{a_1}} \cdot \frac{2^\mu}{2^\mu - 1} \cdot \frac{p_2^{a_2} + \cdots + p_2 + 1}{p_2^{a_2}}.$$

Now $2^{a_1+1} - 1$ necessarily contains the factor $p_2^{a_2}$. It may also contain the factor $2^\mu - 1$ (which is a prime, p_3), which it does in case $p_2^{a_2} + \cdots + p_2 + 1$ is a power of 2. Hence,

$$p_2^{a_2} = 2^{a_1+1} - 1, \quad \text{or} \quad = (2^\mu - 1)^{-1}(2^{a_1+1} - 1).$$

In the former case, we should also have

$$p_2^{a_2} + \cdots + p_2 + 1 = 2^{a_1+1-\mu}(2^\mu - 1) = 2^{a_1+1} - 2^{a_1+1-\mu}.$$

The two equations are mutually exclusive. Hence we have left to consider the case

$$p_2^{a_2} = (2^\mu - 1)^{-1}(2^{a_1+1} - 1) = 2^{a_1+1-\mu} + \frac{2^{a_1+1-\mu} - 1}{2^\mu - 1}.$$

This case also requires

$$p_2^{a_2} + \cdots + p_2 + 1 = 2^{a_1+1-\mu}.$$

The last two equations are mutually exclusive.

We must therefore consider the other case, that in which $1 + p_3^2 + p_3^4 + \cdots + p_3^{a_3-1}$ contains an even number of terms. The expression is then divisible by $1 + p_3^2$, which introduces a third prime; that is, p_2. The result of dividing out this factor is $1 + p_3^4 + p_3^8 + \cdots$. If this were divisible by $1 + p_3^4$ a fourth prime would be introduced; for, $1 + p_3^4$ is prime to p_3 and contains only the factor 2 in common with $p_3^2 + 1$, and is moreover not a power of 2. This fourth prime is inadmissible. Hence, $1 + p_3^4 + p_3^8 + \cdots$ must contain an odd number of terms; for otherwise it is divisible by $1 + p_3^4$. If this number of terms is greater than 1, the expression may be separated into two odd factors, each prime to p_3, introducing at least one other prime. (Compare similar case above.) This is inadmissible; hence, the expression can contain only one term, and this is therefore equal to 1. Hence, $a_3 \leqq 3$; and, therefore, since it is odd and the value 1 has already been excluded, $a_3 = 3$, when $p_3 + 1$ is a power of 2.

Substituting in equation (10) the values $a_3 = 3$, $p_3 + 1 = 2^\mu$, we have

$$2 = \frac{2^{a_1+1} - 1}{2^{a_1}} \cdot \frac{1 + p_2 + \cdots + p_2^{a_2}}{p_2^{a_2}} \cdot \frac{2^{\mu+1}(2^{2\mu-1} - 2^\mu + 1)}{(2^\mu - 1)^3}.$$

Now, if a_2 is even, $1 + p_2 + \cdots + p_2^{a_2}$ is odd and $a_1 = \mu$; hence,

$$1 + p_2 + \cdots + p_2^{a_2} = (2^\mu - 1)^3,$$

and therefore $p_2^{a_2} = (2^{\mu+1} - 1)(2^{2\mu-1} - 2^\mu + 1)$. Expanding these two equations, we have respectively

$$1 + p_2 + \cdots + p_2^{a_2} = 2^{3\mu} - 3 \cdot 2^{2\mu} + 3 \cdot 2^\mu - 1;$$

$$p_2^{a_2} = 2^{3\mu} - 2^{2\mu+1} - 2^{2\mu-1} + 2^{\mu+1} + 2^\mu - 1.$$

By the former of the last two equations $\mu > 1$; for such values of μ the two equations become mutually exclusive. Hence, when $p_3 + 1$ is a power of 2, a_2 is odd.

Now, if $p_2 + 1$ is a power of 2, $= 2^\lambda$, then $a_2 = 3$, by the same reasoning as above for a_3. For this case we have from equation (10),

$$(13) \qquad 2 = \frac{2^{a_1+1} - 1}{2^{a_1}} \cdot \frac{2^{\lambda+1}(2^{2\lambda-1} - 2^\lambda + 1)}{(2^\lambda - 1)^3} \cdot \frac{2^{\mu+1}(2^{2\mu-1} - 2^\mu + 1)}{(2^\mu - 1)^3}.$$

By clearing of fractions and reducing it is easily shown that this equation can not be satisfied in any case. Hence, $p_3 + 1$ and $p_2 + 1$ are not both powers of 2.

If we supposed either $p_2 + 1$ or $p_3 + 1$ to be not a power of 2, a course of reasoning very similar to that by which, in case $p_3 + 1$ is a power of 2, it was shown above that $a_3 = 3$, would bring us to the conclusion that the corresponding a equals 1, provided that that a is odd.

Suppose now that $p_2 + 1$ is not a power of 2. Also let $p_3 + 1 = 2^\mu$ as before. Then, a_2 is odd, as seen above; and is hence $= 1$. Substituting now in equation (10), we have

$$(14) \qquad 2 = \frac{2^{a_1+1} - 1}{2^{a_1}} \cdot \frac{p_2 + 1}{p_2} \cdot \frac{2^{\mu+1}(2^{2\mu-1} - 2^\mu + 1)}{(2^\mu - 1)^3}.$$

whence, necessarily, $p_2 = 2^{2\mu-1} - 2^\mu + 1$. Writing this value for p_2 in equation (14) and simplifying, we have

$$(15) \qquad (2^\mu - 1)^3 = (2^{\mu+2} - 1)(2^{2\mu-2} - 2^{\mu-1} + 1),$$

since, by (14), $a_1 = \mu + 1$. No value of μ satisfies equation (15).

We have now examined every possible case in which $p_3 + 1$ is a power of 2, and a_3 is odd, and find no numbers of the type in consideration; similarly we may completely dispose of the case in which $p_2 + 1$ is a power of 2 and a_2 odd; and that without assuming in this case that a_3 is odd. And since necessarily a_2 or a_3 is odd, this effectually disposes of every case in which either $p_2 + 1$ or $p_3 + 1$ is a power of 2, unless it can happen that one a is even while the corresponding $p + 1$ is a power of 2, a case which is shown below to be impossible. Hence, we have left only the case in which neither $p_2 + 1$ nor $p_3 + 1$ is a power of 2. As already seen, a_2 or a_3 must be odd. We may, without loss of generality, take a_3 to be odd. Then, as we have seen above, $a_3 = 1$. Now, suppose a_2 to be even; say $a_2 = 2v$. Substituting these values in equation (10) we have

$$2 = \frac{2^{a_1+1} - 1}{2^{a_1}} \cdot \frac{p_2^{2v} + p_2^{2v-1} + \cdots + p_2 + 1}{p_2^{2v}} \cdot \frac{p_3 + 1}{p_3}.$$

Since $p_3 + 1$ is not a power of 2, it must contain an odd factor. It can contain no odd prime but p_2. Hence, $p_3 + 1 = 2^x p_2^y$, where $x > 0$, $y > 0$. Also, $y < 2v$; for, to satisfy the equation above we also require $2^{a_1+1} - 1 =$ some power of p_2. Also $p_3 = p_2^{2v} + \cdots + p_2 + 1$, for this last member does not contain 2. These values for $p_3 + 1$ and p_3 are mutually exclusive. Hence, in this case also, a_2 is odd. Hence, as for the case a_2 odd, $p_2 + 1$ is not a power of 2; and, therefore, by previous considerations, $a_2 = 1$. (This also shows to be impossible the case above supposed; namely, one a even and the corresponding $p+1$ a power of 2; for it is now seen that since one a is necessarily odd and its corresponding $p+1$ not a power of 2, the other a is odd also. Of course a here refers to either a_2 or a_3, but not to a_1.)

Substituting in equation (10) the values $a_2 = 1$, $a_3 = 1$, above developed, we have

$$2 = \frac{2^{a_1+1}-1}{2^{a_1}} \cdot \frac{p_2+1}{p_2} \cdot \frac{p_3+1}{p_3}. \tag{16}$$

Now, in this case, neither p_2+1 nor p_3+1 is a power of 2. Hence, the numerator has three odd factors, which may or may not be different. In either case these three factors can not be divided out by two odd primes. This case therefore yields no numbers of the type in consideration.

This completes the investigation of the several cases of our last theorem. We are now able to state in conclusion the following general proposition, all cases having been examined either in this paper or in those already referred to:—

There are but two multiply perfect numbers of three different primes; and these two are $2^3 \cdot 3 \cdot 5$ *and* $2^5 \cdot 3 \cdot 7$. *The multiplicity of each is* 3.

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